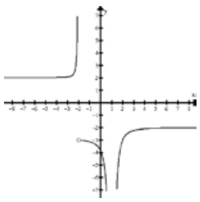
Math 1A Midterm 1 Review Answers

Complete solutions are shown for all questions except those marked ✪.
The missing work for those questions is strictly numeric or algebraic.

[1] ✪ $\lim\limits_{x \to 0} \dfrac{\sqrt{x+\sqrt{\cos x}}-1}{x} = \dfrac{1}{2}$

[2] $\dfrac{f(5)-f(1)}{5-1} = \dfrac{-25-(-1)}{5-1} = -6$ meters per second

[3]



[4] Since $-1 \le \cos\dfrac{1}{x^2} \le 1$ for all x,

therefore $-x^4 \le x^4 \cos\dfrac{1}{x^2} \le x^4$ for all x.

$\lim\limits_{x \to 0}(-x^4) = \lim\limits_{x \to 0} x^4 = 0$.

So, by the Squeeze Theorem, $\lim\limits_{x \to 0} x^4 \cos\dfrac{1}{x^2} = 0$.

[5] [a] $\lim\limits_{x \to -2}(2x-3) = -7$

[b] $\lim\limits_{x \to -1^-}(2x-3) = -5$ and $\lim\limits_{x \to -1^+}(x^2-6) = -5$, so $\lim\limits_{x \to -1} f(x) = -5$

[c] $\lim\limits_{x \to 2^-}(x^2-6) = -2$ and $\lim\limits_{x \to 2^+}(4x-6) = 2$, so $\lim\limits_{x \to 2} f(x)$ DNE

[6] Since $\lim\limits_{x \to 2}(x-2)$ exists (equals 0), $\quad \lim\limits_{x \to 2} \dfrac{\sqrt{x^2+a}-1}{x-2} \lim\limits_{x \to 2}(x-2) = 2\lim\limits_{x \to 2}(x-2)$

Since $\lim\limits_{x \to 2} \dfrac{\sqrt{x^2+a}-1}{x-2}$ and $\lim\limits_{x \to 2}(x-2)$ both exist (given & above), $\quad \lim\limits_{x \to 2} \dfrac{\sqrt{x^2+a}-1}{x-2}(x-2) = 0$

$\lim\limits_{x \to 2}(\sqrt{x^2+a}-1) = 0$

Since $\lim\limits_{x \to 2} 1$ exists (equals 1), $\quad \lim\limits_{x \to 2}(\sqrt{x^2+a}-1) + \lim\limits_{x \to 2} 1 = 0 + \lim\limits_{x \to 2} 1$

Since $\lim\limits_{x \to 2}(\sqrt{x^2+a}-1)$ and $\lim\limits_{x \to 2} 1$ both exist (above), $\quad \lim\limits_{x \to 2}(\sqrt{x^2+a}-1+1) = 1$

$\lim\limits_{x \to 2} \sqrt{x^2+a} = 1$

$\left(\lim\limits_{x \to 2} \sqrt{x^2+a}\right)^2 = 1^2$

Since $\lim\limits_{x \to 2} \sqrt{x^2+a}$ exists (above), $\quad \lim\limits_{x \to 2} \sqrt{x^2+a}\sqrt{x^2+a} = 1$

$\lim\limits_{x \to 2}(x^2+a) = 1$

$4 + a = 1$

$a = -3$

[7] $\lim\limits_{x\to 2}\dfrac{x^2 g(x)}{1+f(x)} = \dfrac{\lim\limits_{x\to 2} x^2 g(x)}{\lim\limits_{x\to 2}(1+f(x))} = \dfrac{\lim\limits_{x\to 2} x\cdot\lim\limits_{x\to 2} x\cdot\lim\limits_{x\to 2} g(x)}{\lim\limits_{x\to 2} 1+\lim\limits_{x\to 2} f(x)} = \dfrac{2\cdot 2\cdot 4}{1+(-3)} = -8$

[8] discontinuities where $x^2-9=0$, ie. at $x=-3$ and $x=3$

$\lim\limits_{x\to -3^-} f(x) = -\infty \quad \left(\dfrac{-1}{0^+}\right)$ $\lim\limits_{x\to -3^+} f(x) = \infty \quad \left(\dfrac{-1}{0^-}\right)$ $\lim\limits_{x\to 3^-} f(x) = -\infty \quad \left(\dfrac{5}{0^-}\right)$ $\lim\limits_{x\to 3^+} f(x) = \infty \quad \left(\dfrac{5}{0^+}\right)$

[9] [a] Since $f(-1)$ DNE, there is no such a

[b] $\lim\limits_{x\to 2^-}(3-x)=1$ and $\lim\limits_{x\to 2^+}(bx-1)=2b-1$, so $\lim\limits_{x\to 2} f(x)$ exists only if $2b-1=1$ ie. $b=1$

★ It was not stated that you need to check that f is continuous at $x = 2$ with this value of b, but it is strongly recommended, to be sure the answer isn't that there is no such b

[c] $\lim\limits_{x\to -1^-}(2x+6)=4$ and $\lim\limits_{x\to -1^+}(3-x)=4$, so $\lim\limits_{x\to -1} f(x)$ exists and $\lim\limits_{x\to -1} f(x)=4$ but $f(-1)$ DNE,

so $x=-1$ is a removable discontinuity

$\lim\limits_{x\to 2^-}(3-x)=1$ and $\lim\limits_{x\to 2^+}(3x-1)=5$, so both one-sided limits exist but are not equal,

so $x=2$ is a jump discontinuity

[10] Let $f(x)=\cos 2x - x^2$.

Since $\cos 2x$ (a continuous trigonometric function composed with a polynomial function)

and x^2 (a polynomial function) are both continuous for all x,

so is their difference $f(x)=\cos 2x - x^2$.

Since $f(\pi)=1-\pi^2<0<1=f(0)$,

by the Intermediate Value Theorem, there is a value c in the interval $(0,\pi)$ such that $f(c)=\cos 2c - c^2 = 0$, ie. $\cos 2c = c^2$.

So the equation $\cos 2x = x^2$ has a solution in the interval $[0,\pi]$.

[11] $2x-1=0 \Rightarrow x=\dfrac{1}{2}$ and $\lim\limits_{x\to \frac{1}{2}^+}\dfrac{\sqrt{4+9x^2}}{2x-1}=\infty \quad \left(\dfrac{\frac{5}{2}}{0^+}\right)$

$$\lim_{x\to -\infty}\frac{\sqrt{4+9x^2}}{2x-1} = \lim_{x\to -\infty}\frac{\sqrt{4+9x^2}}{2x-1}\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to -\infty}\frac{\sqrt{4+9x^2}}{2x-1}\frac{-\sqrt{\frac{1}{x^2}}}{\frac{1}{x}} = \lim_{x\to -\infty}\frac{-\sqrt{\frac{4}{x^2}+9}}{2-\frac{1}{x}} = \frac{-\sqrt{0+9}}{2-0} = -\frac{3}{2}$$

$$\lim_{x\to \infty}\frac{\sqrt{4+9x^2}}{2x-1} = \lim_{x\to \infty}\frac{\sqrt{4+9x^2}}{2x-1}\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to \infty}\frac{\sqrt{4+9x^2}}{2x-1}\frac{\sqrt{\frac{1}{x^2}}}{\frac{1}{x}} = \lim_{x\to \infty}\frac{\sqrt{\frac{4}{x^2}+9}}{2-\frac{1}{x}} = \frac{\sqrt{0+9}}{2-0} = \frac{3}{2}$$

Vertical asymptote: $x=\dfrac{1}{2}$

Horizontal asymptotes: $y=\pm\dfrac{3}{2}$

[12] $f'(-2) = \lim\limits_{b\to -2}\dfrac{f(b)-f(-2)}{b-(-2)} = \lim\limits_{b\to -2}\dfrac{b^3-3b+2}{b+2} = \lim\limits_{b\to -2}\dfrac{(b+2)(x^2-2b+1)}{b+2} = \lim\limits_{b\to -2}(b^2-2b+1) = 9$

$f'(-2) = \lim\limits_{h\to 0}\dfrac{f(-2+h)-f(-2)}{h} = \lim\limits_{x\to -2}\dfrac{(-2+h)^3-3(-2+h)+2}{h} = \lim\limits_{x\to -2}\dfrac{-8+12h-6h^2+h^3+6-3h+2}{h}$

$= \lim\limits_{x\to -2}\dfrac{9h-6h^2+h^3}{h} = \lim\limits_{x\to -2}(9-6h+h^2) = 9$

[13] [a] $f(x)=\cos \pi x$, $a=-1$ [b] $f(x)=x^2-x$, $a=-2$

[14] ✪ 1.5 feet per minute

[15] ✪ $y+4=2(x-2)$

[16] $f'(-2) < f'(4) < 0 < f'(2) < f'(-4)$

[17] [a] If the refrigerator temperature is $4^{\circ}C$, the food will defrost in 6 hours.

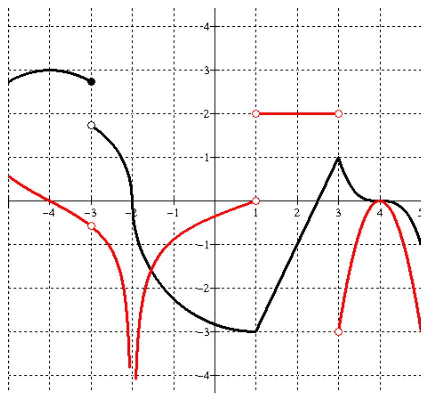
[b] If the refrigerator temperature is $4^{\circ}C$, the food will defrost 1 hour sooner for each $1^{\circ}C$ increase in the refrigerator's temperature.

[c] No. The defrost time should always decrease if the refrigerator temperature increases. The frozen food will always defrost faster in a warmer refrigerator.

[18] ✪ [a] $f'(t)=\dfrac{1}{2(1-t)^{\frac{3}{2}}}$ [b] $g'(x)=\dfrac{8}{(2-x)^2}$

[19] [a] $x=-3$ (discontinuous)
$x=-2$ (vertical tangent line)
$x=1, 3$ (cusps)

[b]



[20] Since the line $x-2y=6$ (ie. $y=\frac{1}{2}x-3$) is tangent to $y=f(x)$ at $x=4$,
therefore the point of tangency is $(4, \frac{1}{2}(4)-3)$ or $(4,-1)$.
That means $f(4)=-1$ and $f'(4)=\frac{1}{2}$.
Since $f'(4)$ exists, therefore f is differentiable at $x=4$ (by the definition of "differentiable").
Since f is differentiable at $x=4$, therefore f is continuous at $x=4$ (by the "differentiability implies continuity" theorem).
Since f is continuous at $x=4$, therefore $\lim_{x\to 4} f(x)=f(4)=-1$ (by the definition of "continuous at a point").